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1. Introduction

The wave profile generated by an obstacle moving at constant velocity U over a water surface of infinite extent appears to be stationary with respect to the moving body provided, of course, the motion has been maintained for a long time. When the gravitational and capillary effects are both taken into account, the surface waves so generated may possess a minimum phase velocity c_m characterized by a certain wave length, say λ_m (see Ref. 1, p. 459). If the velocity U of the solid body is greater than c_m, then the physically correct solution of this two-dimensional problem requires that the gravity waves (of wave length greater than λ_m) should exist only on the downstream side and the capillary waves (of wave length less than λ_m) only on the upstream side.

If one follows strictly the so-called steady-state formulation so that the time does not appear in the problem, one finds in general that it is not possible to characterize uniquely the mathematical solution with the desired physical properties by imposing only the boundedness conditions at infinity.* Some stronger radiation conditions are actually necessary. In the linearized treatment of this stationary problem, several methods[1-12] have been employed, most of which are aimed at obtaining the correct solution by introducing some artificial device, either of a mathematical or physical nature. One of these methods widely used was due to Rayleigh[2], and was further discussed by Lamb.[1] In the analysis of this problem Rayleigh introduced a "small dissipative force", proportional to the velocity relative to the moving stream. This

* In the case of a three-dimensional steady-state problem, even the condition that the disturbance should vanish at infinity is not sufficient to characterize the unique solution.

"law" of friction does not originate from viscosity and is hence physically fictitious, for in the final result this dissipation factor is made to vanish eventually. In the present investigation, Rayleigh's friction coefficient is shown to correspond roughly to a time convergence factor for obtaining the steady-state solution from an initial value problem. (It is not a space-limit factor for fixing the boundary conditions at space infinity, as has usually been assumed in explanation of its effect). Thus, the introduction of Rayleigh's coefficient is only a mathematical device to render the steady-state solution mathematically determinate and physically acceptable. For a physical understanding, however, it is confusing and even misleading; for example, in an unsteady flow case it leads to an incomplete solution, as has been shown by Green.[3] Another approach, purely of a physical nature, was used by Michell[4] in his treatment of the velocity potential for thin ships. To make the problem determinate, he chose the solution which represents the gravity waves propagating only downstream and discarded the part corresponding to the waves traveling upstream. For two-dimensional problems with the capillary effect, this method would mean a superposition of simple waves so as to make the solution physically correct. Some other methods appear to be limited in the necessity of interpreting the principal value of a certain kind of improper integral. In short, as to their physical soundness and mathematical rigor, or even to their merits or demerits, the preference of one method over the others has remained nevertheless a matter of considerable dispute. Only until recently the steady-state problem has been treated by first formulating a corresponding initial value problem.[3, 12] A brief historical sketch of these methods is given in the next section.

The purpose of this paper is to try to understand the physical mechanism underlying the steady configuration of the surface wave phenomena and to clarify to a certain extent the background of the artifices adopted for solution of steady-state problems. The point of view to be presented here is that this problem should be formulated first as an initial value problem (for example, the body starts to move with constant velocity at a certain time instant), and then the stationary state is sought by passing to the limit as the time tends to infinity. If at any finite time instant the boundary condition that the disturbance vanishes at infinity (because of the finite wave velocity) is imposed, then the limiting solution as the time tends to infinity is determinate and bears automatically the desired physical properties. Also, from the integral representation of the linearized solution, the asymptotic behavior of the wave form for large time is derived in detail, showing the distribution of the wave trains in space. This asymptotic solution exhibits an interesting picture which reveals how the dispersion* generates two monochromatic wave trains, with the capillary wave in front of, and the gravity wave behind, the surface pressure. The special cases $U < c_m$ and $U = c_m$ are also discussed. The viscous effect and the effect of superposition are commented upon later. Through this detailed investigation it is found that the dispersive effect, not the viscous effect plays the significant role in producing the final stationary wave configuration.

2. Historical Sketch of Existing Theories

In most of the early works on the steady-state surface wave problems there seem to be two major lines of thought in regard to obtaining the mathematical solution in agreement with physical observations. When the viscosity is neglected in the formulation, the integral representation of the wave profile contains an improper integral of the form

$$\int_0^\infty \frac{\cos kx}{k - \kappa}\, dk\ , \qquad \kappa \text{ being a positive constant.}$$

If the integral above is interpreted as its Cauchy principal value, then the solution shows that there are gravity waves upstream as well as downstream, a fact certainly in contradiction with physical experience. To improve the result, one approach assumes that some important

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* By dispersive medium is meant one in which the wave velocity of a propagating wave depends on the wave length, so that a number of wave trains of different wave lengths tends to form groups, propagating with group velocities which are in general different from the phase velocities of individual wave trains. In case of waves on the water surface, both the gravity and surface tension are responsible for dispersion.

physical factors (such as viscosity) are being omitted in the formulation, while the second approach contends that the interpretation of such integrals is only a mathematical matter.

To avoid this indeterminateness, Rayleigh[2] introduced a fictitious small dissipative force, proportional to the relative velocity so that the irrotational character of the motion is preserved. With the introduction of this force, the factor $(k-\kappa)$ in the integrand becomes $(k-\kappa-i\mu)$ where $i=\sqrt{-1}$ and μ is Rayleigh's friction coefficient (>0); the problem is then determinate since the singularity is no longer on the path of integration. This artifice has been widely used. Some phases of the argument against this device were mentioned in the previous section.

Another method is based on the following physical argument. The phenomenon of group velocity (denoted by c_g) arises from the mechanism of dispersion; it has the important physical significance that the wave energy is transmitted at the rate c_g even if the wave is monochromatic. It is known (Ref. 1, p. 460) that for long gravity waves, $c_g = \frac{1}{2}c$ (c being the phase velocity of the wave), while for short capillary waves, $c_g = \frac{3}{2}c$. Hence it follows that in steady flows the *capillary waves cannot trail and the gravity waves cannot precede the* surface force. Therefore, one may always add to the formal solution an endless train of free waves of appropriate wave length and amplitude such that this physical requirement is fulfilled. Michell's argument[4] is of this nature; another example is Ref. 5.

Hogner[6] has treated the three-dimensional case of a moving pressure disturbance on the water surface, without the use of Rayleigh's dissipative force, by making a choice of the integration path, which was only justified by the final result.

Dean[7] has treated the two-dimensional case of gravity waves generated by a finite band of surface pressure by using complex variable techniques. To make the motion definite, it is assumed that the far upstream region is not disturbed and the path of integration in the complex plane is adjusted accordingly. The matter would be complicated if the capillary effect were included. Besides this method cannot be generalized to the three-dimensional case. A more general discussion of using a complex variable to treat the wave problem was given by H. Lewy.[8]

Peters[9] has presented anew a general approach to both the two-dimensional and three-dimensional problems, but in his treatment the integral of the above form was evaluated by following a path indented at $k = \kappa$, the indentation being so chosen as to make the disturbance vanish far upstream. This method, in the large, is then equivalent to using Rayleigh's factor only in the final interpretation of the integral.

In 1953 Timman and Vossers[10] presented a new method for solving the gravity wave problem by applying the complex Fourier transform without using Rayleigh's coefficient. They point out that, even if the gravity wave exists only on the downstream side, it is the complex Fourier transform which should be employed in the solution of the steady-state problem. This method would be complicated with the addition of surface tension since the assumption should also be made that the capillary waves vanish far downstream.

Aside from the supposition that the steady-state problem can be a natural formulation provided some remedy for the difficulties is found, it may be pointed out that the detailed mechanism of dispersion can only be seen from the result of a corresponding initial value problem. This idea, apparently already conceived in Kelvin's famous work[11], was first executed (in 1948) by Green[3] to treat two problems: Lamb's radiation problem of the oscillating surface pressure disturbance and the problem of the uniformly moving disturbance. The radiation condition for Green's first problem was also discussed recently by Stoker.[12] Incidentally, this idea was also mentioned, but was not carried out, by Havelock.[13]

3. Linearized Formulation of the Problem

The problem in question concerns the propagation of surface waves in an infinite ocean, initially at rest, due to a pressure distribution which moves with constant rectilinear velocity U over the free surface. The liquid medium is taken to be inviscid and incompressible of constant density ρ. The resulting flow is assumed to be irrotational so that a velocity potential exists in the medium. It is further assumed that the motion is a small perturbation; thus a linear theory can be applied. We restrict ourselves to the two-dimensional motion in an xy-plane, with the y-axis pointing vertically upward, $y=0$ coinciding with the free surface at rest, and the origin being fixed with respect to the applied pressure p_o so that the flow at infinity has a uniform velocity U in the direction of x positive. In this coordinate system, the surface pressure may be described more precisely as

$$\begin{aligned} p_o &= 0 \qquad \text{for time} \quad t \leq 0 \ , \\ &= p_o(x) \qquad \text{for} \quad t > 0 \ , \end{aligned} \tag{1a}$$

where $p_o(x)$ is an arbitrary function of x and is assumed to be subject to the condition

$$\int_{-\infty}^{\infty} \left| p_o(x) \right| \, dx < \infty \ . \tag{1b}$$

A perturbed velocity potential $\varphi(x, y, t)$ is defined for $y \leq 0$ and $t > 0$ so that the total flow velocity has the components $(U + \varphi_x, \varphi_y)$ where the subscripts denote the partial derivatives. The velocity potential, of course, satisfies the relation

$$\varphi_{xx} + \varphi_{yy} = 0 \qquad \text{for} \quad y < 0, \quad t > 0 \ . \tag{2}$$

When both gravity and surface tension are considered, the linearized boundary conditions on the free surface can be shown (for example, by generalizing the argument of Ref. 1, p. 363 and p. 456) to be

$$\left.\begin{aligned} &\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}\right) \zeta = \varphi_y && (3) \\ &\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}\right) \varphi + \left(g - \sigma \frac{\partial^2}{\partial x^2}\right) \zeta = -\frac{1}{\rho} p_o(x) && (4) \end{aligned}\right\} \text{ on } y=0 \text{ for } t > 0 ,$$

where $\zeta(x, t)$ is the vertical displacement of the free surface, measured upward from $y=0$, g is the gravitational constant, $\sigma = \Sigma/\rho$, and Σ is the surface tension of the liquid-air interface. We further require that for any finite t,

$$\zeta, \ \zeta_x \ \text{ be absolutely integrable with respect to } x; \tag{5}$$

and

$$\varphi, \ |\text{grad } \varphi| \ \text{ be absolutely integrable with respect to } x \text{ for any fixed } y \leq 0 . \tag{6}$$

These conditions are suggested by the argument that the wave velocity is finite for waves having finite wave length; thus, for any finite t, one can always find a large enough distance x such that beyond this point the wave disturbance has not been registered. At time $t=0$, we prescribe the zero initial conditions

$$\varphi(x, y, 0) = \zeta(x, 0) = 0 , \qquad y \leq 0 . \tag{7}$$

As a remark, it can be seen by integrating Eqs. (3), (4) from $t=0$ to some small value that $\varphi(x, 0, t) = 0(t)$ as $t \to 0$ and $\zeta_t(x, 0) = 0$. In particular, $\varphi_t(x, 0, 0) = - p_o(x)/\rho$. Note that no assumptions are made regarding the behavior of the solution as $t \to \infty$, but the limiting solution is to be accepted as a natural consequence.

Next we introduce the Laplace transform with respect to t and Fourier transform with respect to x, of a function $f(x, t)$, as defined by

$$\bar{f}(x, s) = \int_0^{\infty} e^{-st} f(x, t) \, dt ,$$

$$\tilde{f}(k, t) = \int_{-\infty}^{\infty} e^{ikx} f(x, t) \, dx .$$

Conditions (1b) and (3)-(6) imply that the Laplace and Fourier transform of φ and ζ exist (for the general condition of existence, see e.g., Ref. 14). The transform of Eq. (2) under condition (6) is

$$\tilde{\bar{\varphi}}_{yy} - k^2 \tilde{\bar{\varphi}} = 0$$

which has the solution of the form

$$\tilde{\tilde{\varphi}} = A(s,k)\, e^{|k|y} \qquad \text{for} \quad y \leq 0\ , \tag{8}$$

where $A(s,k)$ represents an arbitrary function of s and k. By making use of this result and the transform of (3) and (4), together with conditions (5)-(7), one obtains

$$\tilde{\tilde{\zeta}} = -\frac{1}{\rho s}\ \tilde{p}_o(k) \left[g + \sigma k^2 + (s - ikU)^2/|k| \right]^{-1} \tag{9}$$

and

$$A(s,k) = (s - ikU)\ \tilde{\tilde{\zeta}}/|k|\ . \tag{10}$$

Finally, application of the inverse transforms (their formulas are given elsewhere, e.g., see Ref. 14) to Eq. (9) yields the following integral representation of ζ,

$$\zeta(x,t) = \frac{1}{\rho} \int_{-\infty}^{\infty} H(x-\xi, t)\ p_o(\xi)\ d\xi \tag{11}$$

with

$$H(x,t) = -\frac{1}{2\pi^2 i} \int_\Gamma e^{st} \frac{ds}{s} \int_0^\infty \frac{(\Lambda + s^2)\cos kx + 2Uks \sin kx}{(\Lambda + s^2)^2 + (2Uks)^2}\ k\,dk\ , \tag{12}$$

and

$$\Lambda(k) \equiv k(g + \sigma k^2 - U^2 k)\ . \tag{13}$$

In (12), the contour Γ is taken parallel to the imaginary s-axis, located to the right of all the singularities of the integrand in the complex s-plane. Note that $H(x,t)$ is the singular solution of ζ when $p_o(x)$ becomes the Dirac delta function, $p_o(x) = \rho\,\delta(x)$; $H(x,t)$ is therefore a typical solution which exhibits all the important features of this problem and will thus be called the fundamental solution of the wave profile.

If the variable k in the Fourier integral of Eq. (12) is taken to be real, then the singularities of the integrand in the s-plane are five simple poles at

$$s = 0, \ \pm i\,(\omega \pm kU) \ , \tag{14}$$

where

$$\omega = \left[k(g+\sigma k^2)\right]^{1/2} = \left[\Lambda + U^2 k^2\right]^{1/2} \ . \tag{15}$$

Hence the contour Γ may be taken along $s = s_o + i\eta$ with $s_o = \text{const} > 0$ and η running from $-\infty$ to $+\infty$.

Also, since the real parts of the poles all vanish, one may foresee that the wave components (with any real wave number k) in the Fourier integral will have no exponential time-damping factor, but are purely oscillatory in nature. For, after carrying out the s-integral, one immediately obtains that $H(x, t) = 0$ for $t < 0$, and for $t \geq 0$,

$$H(x, t) = -\frac{1}{\pi} \int_o^\infty \frac{k}{\omega}\, dk \int_o^t \cos k(x - U\tau)\, \sin \omega\tau \, d\tau \tag{16a}$$

of which the integration on τ yields:

$$H(x, t) = -\frac{1}{\pi} \int_o^\infty \left\{ \frac{\cos kx}{\Lambda} - \frac{\cos\left[k(x-Ut)+\omega t\right]}{2\omega(\omega - Uk)} - \frac{\cos\left[k(x-Ut)-\omega t\right]}{2\omega(\omega+Uk)} \right\} k\,dk \ , \tag{16b}$$

or, after a suitable regrouping,

$$H(x, t) = -\frac{1}{\pi} \int_o^\infty \left\{ \frac{\sin \frac{\omega - Uk}{2} t}{(\omega - Uk)} \sin\left[kx + \frac{\omega - Uk}{2} t\right] - \frac{\sin \frac{\omega+Uk}{2} t}{(\omega+Uk)} \sin\left[kx - \frac{\omega+Uk}{2} t\right]\right\} \frac{k}{\omega}\, dk \ . \tag{16c}$$

These integral representations of H may serve as alternatives of Eq. (12). In these equations, $\Lambda(k)$ and $\omega(k)$ are given by Eq. (15) which may also be written

$$\Lambda(k) = (\omega - Uk)(\omega + Uk) = \sigma k(k - \kappa_1)(k - \kappa_2) \tag{17}$$

where

$$\left.\begin{matrix} \kappa_1/\kappa_m \\ \kappa_2/\kappa_m \end{matrix}\right\} = \left(\frac{U}{c_m}\right)^2 \mp \left[\left(\frac{U}{c_m}\right)^4 - 1\right]^{1/2}, \quad \kappa_m = \left(\frac{g}{\sigma}\right)^{1/2},$$

$$c_m^{\ 2} = 2(g\sigma)^{1/2} . \tag{18}$$

Thus, the two zeros of Λ/k, namely, κ_1 and κ_2, correspond to the zeros of $(\omega - Uk)$ on account of Eq. (17). These two zeros are (i) real and unequal $(\kappa_1 < \kappa_m < \kappa_2)$ for $U > c_m$; (ii) real and equal $(\kappa_1 = \kappa_2 = \kappa_m)$ for $U = c_m$ and (iii) complex conjugate $(\kappa_2 = \bar{\kappa}_1)$ for $U < c_m$. For a water surface, $c_m = 23.2$ cm/sec; $\kappa_m = 3.63$/cm with the corresponding wave length $\lambda_m = 2\pi/\kappa_m = 1.73$ cm. We shall first consider case (i), while cases (ii) and (iii) will be deferred to Sec. 6 and 7.

Consider k to be a real variable for the moment, then the integrand in Eq. (16c) is seen to be a continuous function of k, x and t since the singularities at $k = \kappa_1$ and κ_2 for $U \geq c_m$ are actually removable. Furthermore, the integral converges uniformly for any finite closed interval of x and t. The same assertion holds true for the integral in Eq. (16b) if the integrand is considered as a whole. However, when the first two terms in Eq. (16b) are integrated separately along the real k-axis, the integrals may be interpreted as their respective Cauchy principal value (the last term is regular on the positive k-axis).

If, on the other hand, k is taken to be a complex variable, then $\omega(k)$ has three branch points at $k = 0$, $\pm i\kappa_m$. If three branch cuts are introduced, cut from the branch points to the negative infinity along paths parallel to the real axis (see Fig. 1), then ω, and hence also the integrand in Eq. (16), will be an analytic function of k, regular in the cut plane. Consequently, to calculate the integral in Eq. (16) one may either use the original path along the positive real axis, or deform the path into a new contour as long as the integral converges (the path L shown in Fig. 1 will be discussed in the next section). It should be pointed out that the solution obtained here is also seen to be the unique solution of our problem because under condition (7), $\zeta(x,t) \equiv 0$ for $t > 0$ if $p_o(x) \equiv 0$.

In Eq. (16b), the first term is an even function of x, independent of t, and is the formal solution of the steady-state formulation, obtained as usual by omitting ζ_t and φ_t in Eqs. (3) and (4) at the beginning. The second and last term in the integrand represent waves propagating away from the point $x = Ut$ towards the upstream and downstream respectively. Each elementary wave of wave number k propagates with the phase velocity

$$c = \frac{\omega}{k} = \left(\frac{g}{k} + \sigma k\right)^{1/2} = \frac{c_m}{\sqrt{2}} \left(\frac{k}{\kappa_m} + \frac{\kappa_m}{k}\right)^{1/2} \tag{19}$$

relative to the point $x = Ut$ (or the fluid at rest); while all the components of wave number in k and $k + dk$ form a wave group, propagating with the group velocity

$$c_g = \frac{d\omega}{dk} = \frac{c_m}{\sqrt{8}} \left(3\frac{k}{\kappa_m} + \frac{\kappa_m}{k}\right)\left(\frac{k}{\kappa_m} + \frac{\kappa_m}{k}\right)^{-1/2}, \tag{20}$$

also relative to $x = Ut$. The integration with respect to k then leads to a resultant dispersion since ω depends on k. It is thus obvious that, at least for finite t, the surface wave H on the $x > 0$ side will be different from that on the $x < 0$ side since the first term of (16b) is even in x while the last two terms are not.

4. The Limiting Solution as $t \to \infty$, $U > c_m$

In this section the limit of $H(x,t)$ as $t \to \infty$ for $U > c_m$ will first be calculated by applying the Tauberian theorem (see Ref. 14, p. 187): In the relation

$$F(t) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} e^{st}\, G(s)\, \frac{ds}{s}$$

let $G(s)$ be an analytic function of s, regular in the half-plane $R\ell\ s \geq s_o > 0$, and $b \geq s_o$, and let

$$\lim_{s\to 0+} G(s) = A\,, \qquad A \text{ being any constant.} \tag{21a}$$

Then

$$\lim_{t\to +\infty} F(t) = A \tag{21b}$$

if, and only if

$$\frac{1}{t}\int_o^t \tau\, \frac{dF}{d\tau}\, d\tau \to 0 \quad \text{as} \quad t \to +\infty\,. \tag{22}$$

Applying this theorem to our problem, one may take $H(x,t)$ of Eq. (12) to be $F(t)$ for any fixed x so that

$$G(s,x) = -\frac{1}{\pi}\int_o^\infty \frac{(\Lambda+s^2)\cos kx + (2U ks)\sin kx}{(\Lambda+s^2)^2 + (2U ks)^2}\, kdk \tag{23}$$

whereas the constant A of (21a), if it exists, may now be a function of x.

We shall first evaluate the limit of (23) as $s \to 0+$. In the part of (23) containing $\cos kx$, s may be neglected altogether as $s \to 0+$, Thus

$$A_1(x) \equiv -\frac{1}{\pi}\lim_{s\to 0+}\int_o^\infty \frac{(\Lambda+s^2)(\cos kx)\, kdk}{(\Lambda+s^2)^2 + (2U ks)^2} = -\frac{1}{\pi} P\int_o^\infty \frac{\cos kx}{\Lambda}\, kdk$$

$$= \frac{-1}{\pi\sigma(\kappa_2-\kappa_1)}\, R\ell\ P\int_o^\infty \left(\frac{1}{k-\kappa_1} - \frac{1}{k-\kappa_2}\right) e^{ikx}\, dk$$

where $R\ell$ stands for "the real part of", and P, the Cauchy principal value of the integral. To evaluate its principal value, one may regard k to be complex and then construct an appropriate closed contour as follows. The original path along the positive real axis, broken at κ_1 and κ_2, is connected up by two small semicircles of radius ϵ, and is joined by a large semicircle of radius R (in the upper or lower half plane for $x > 0$ or $x < 0$, see Fig. 2) and back to $k = 0$ by the negative real axis. By Cauchy's theorem, the integral along the closed contour is zero. Upon passing to the limit $R \to \infty$, $\epsilon \to 0$, the contribution from the ϵ-circles accounts for half the residues. Finally, one obtains

$$A_1(x) = -\frac{\operatorname{sgn} x}{\sigma(\kappa_2 - \kappa_1)} \left[\sin \kappa_1 x - \sin \kappa_2 x\right] + \frac{1}{\pi\sigma} \int_0^\infty \frac{\cos kx dk}{(k+\kappa_1)(k+\kappa_2)} \tag{24}$$

in which $\operatorname{sgn} x = 1$ for $x > 0$ and -1 for $x < 0$. Thus A_1 is an even function of x, as it is in its original form.

In the second part of Eq. (23), we may replace $(\Lambda + s^2)$ by Λ as $s \to 0$, hence

$$A_2(x) = -\frac{1}{\pi} \lim_{s \to 0+} \int_0^\infty \frac{2Us \sin kx}{\Lambda^2 + (2Uks)^2} k^2 dk .$$

To calculate this limit, the range of integration may be reduced to two short stretches: from $k = \kappa_1 - \epsilon$ to $\kappa_1 + \epsilon$ and from $\kappa_2 - \epsilon$ to $\kappa_2 + \epsilon$ (with $\epsilon > 0$), for outside these two stretches the integrand tends to zero uniformly as $s \to 0$. Moreover, within the short stretches $\sin kx$ may be approximated by $\sin \kappa_1 x$ and $\sin \kappa_2 x$ respectively. Then

$$A_2(x) = -\frac{1}{\pi} (\sin \kappa_1 x + \sin \kappa_2 x) \lim_{s \to 0+} \int_0^\epsilon \frac{(4U s)\, d\tau}{[\sigma(\kappa_2 - \kappa_1)\tau]^2 + (2U s)^2} \quad *$$

$$= -\frac{1}{\sigma(\kappa_2 - \kappa_1)} \left[\sin \kappa_1 x + \sin \kappa_2 x\right] \tag{25}$$

which is an odd function of x. Hence $G(s, x) \to A_1(x) + A_2(x)$ as $s \to 0+$.

IF ONE TAKES LIMIT AS $\epsilon \to 0$ BEFORE THE LIMIT $S \to 0^+$ THEN $A_2(x) = 0$. THAT IS, YOU GET THE WRONG ANSWER.

Next we show that $H(x,t)$ satisfies the necessary and sufficient condition (22.) In fact, we shall keep x fixed at any finite value. From Eq. (16)

$$\frac{\partial H}{\partial t} = -\frac{1}{2\pi} \operatorname{Im} \int_0^\infty \left\{ e^{i(\omega - Uk)t + ikx} + e^{i(\omega + Uk)t - ikx} \right\} \frac{k}{\omega}\, dk \tag{26}$$

which represents a continuous function of x and t, and hence bounded, in any finite range of x and t. Its asymptotic value for large t can be calculated by using the method of stationary phase (see Ref. 16, p. 515). For fixed x but large t, the stationary point of the first exponential function in the integrand is given by $d(\omega - Uk)/dk = 0$, or $\omega'(k) = U$. The basic features of the curve $\omega = \omega(k)$ given by Eq. (15) are shown in Fig. 3. For positive k, $\omega(k)$ is a monotonically increasing function of k, with an inflection point at κ_0 and its slope $\omega'(k) \geq \omega'(\kappa_0)$ where

$$\kappa_0 = (3 + 2\sqrt{3})^{-1/2} \kappa_m, \qquad \omega'(\kappa_0) = \left(\frac{6\sqrt{3} - 9}{4}\right)^{1/4} c_m, \qquad \omega''(\kappa_0) = 0. \tag{27}$$

The ratio ω/k gives the wave velocity c and the slope $\omega'(k)$ gives the group velocity c_g of the wave with wave number k, both velocities being referred to the fluid at rest (see Eqs. (19) and (20).). The line $\omega = Uk$ intercepts the curve $\omega(k)$ at κ_1 and κ_2 for $U > c_m$, is tangential to the curve at κ_m for $U = c_m$ and does not intercept the curve $\omega(k)$ (except at the origin) for $U < c_m$. Thus it is clear that for $U > c_m$ the equation $\omega'(k) = U$ has two real roots, say k_1 and k_2 with $0 < k_1 < \kappa_1$ and $\kappa_0 < k_2 < \kappa_2$ (see Fig. 3); it is unnecessary to obtain the explicit expression for k_1 and k_2 here. In accord with the principle of stationary phase, the most significant contribution to the first integral of Eq. (26) comes from a neighborhood of the stationary points k_1 and k_2. Carrying out this approximation one obtains the asymptotic representation

$$\int_0^{\infty} e^{i(\omega-Uk)t+ikx}\ \frac{k}{\omega}\, dk \cong \frac{1}{\sqrt{t}}\left\{b_1 e^{ia_1t+ik_1x}+b_2 e^{ia_2t+ik_2x}\right\}\left[1+O\left(\frac{1}{\sqrt{t}}\right)\right], \tag{28a}$$

where a_1, a_2, b_1, b_2 are four coefficients, depending only on the value of k, $\omega(k)$ and $\omega''(k)$ at k_1 or k_2 (their exact expression being again unnecessary). The second exponential function in Eq. (26) has, however, no stationary point for $k>0$ since $\omega'(k)+U$ clearly has no real zeros. It then follows that this second integral will be of higher order of small quantities than the first as $t\to\infty$. In fact it can be shown, for example, by integration by parts, that for t large and x fixed,

$$\int_0^{\infty} e^{i(\omega+Uk)t-ikx}\ \frac{k}{\omega}\, dk \cong -4ig^{-2}t^{-3}\left[1+O\left(\frac{1}{t}\right)\right] \quad . \tag{28b}$$

The asymptotic representation of $\partial H/\partial t$ given by Eq. (26) is then obtained by combining the above expressions. Finally, to evaluate the limit (22) the interval can be divided into two parts, from $\tau=0$ to T in which $\partial H/\partial t$ is bounded, say, by B, and from $\tau=T$ to t in which the asymptotic value of $\partial H/\partial t$ is valid and may be represented sufficiently by its typical term $bt^{-1/2}\exp(iat)$ of Eq. (28). Upon integration by parts in the second interval, one obtains for fixed x,

$$\frac{1}{t}\int_0^t \tau\,\frac{\partial H}{\partial \tau}\,d\tau \sim \frac{B}{t}\int_0^T \tau\,d\tau + \frac{b}{t}\int_T^t \tau^{1/2}\, e^{ia\tau}\left[1+0(\tau^{-1/2})\right]d\tau$$

$$\sim 0(t^{-1/2}\,e^{iat}) \tag{29}$$

which tends to zero, as $t\to+\infty$, uniformly for any finite x. Therefore, it follows from the Tauberian theorem that as $t\to+\infty$,
$H(x,t)\to A_1(x)+A_2(x)$, or

$$H \cong \frac{-2}{\sigma(\kappa_2-\kappa_1)} \begin{Bmatrix} \sin \kappa_1 x \\ \sin \kappa_2 x \end{Bmatrix} + \frac{1}{\pi\sigma} \int_0^\infty \frac{\cos kx dk}{(k+\kappa_1)(k+\kappa_2)} \quad \text{for} \quad \begin{cases} x \geq 0 \\ x \leq 0 \end{cases} . \tag{30}$$

An alternative method* to obtain the limit of $H(x, t)$ as $t \to +\infty$ is by deforming the original path of integration to an appropriate contour along which the transient waves represented by the last two terms of (16b) die out as $t \to +\infty$. That the integration path may be deformed is justified by the analyticity of the integrand in Eq. (16b), as was discussed in Sec. 3. To ascertain the behavior of the integrand near $k = \kappa_1$ in the complex k-plane, we let $k = \kappa_1 + z$; then for $|z| << \kappa_1$ we obtain from Eq. (17)

$$\omega - Uk = -\alpha z - \beta_1 z^2 + O(z^3) \tag{31a}$$

with

$$\alpha = \frac{U}{2}\left(\frac{\kappa_2-\kappa_1}{\kappa_2+\kappa_1}\right) > 0 , \qquad \beta_1 = \left(\frac{U}{2}\right)\left[\frac{4\kappa_2^2 - 3(\kappa_2+\kappa_1)^2}{4\kappa_1(\kappa_2+\kappa_1)^2}\right]. \tag{31b}$$

Hence $\left| \exp\left[i(\omega - Uk)t\right] \right| \cong \exp(\alpha t \operatorname{Im} z)$ which diminishes exponentially as $t \to +\infty$ only for $\operatorname{Im} z < 0$. Similarly, if we put $k = \kappa_2 + z$, then for $|z|$ small,

$$\omega - Uk = \alpha z + \beta_2 z^2 + O(z^3) \tag{32a}$$

with α given above and

$$\beta_2 = \left(\frac{U}{2}\right)\left[\frac{3(\kappa_1+\kappa_2)^2 - 4\kappa_1^2}{4\kappa_2(\kappa_1+\kappa_2)^2}\right]. \tag{32b}$$

Thus the same exponential function dies out as $t \to +\infty$ only for $\operatorname{Im} z > 0$ near κ_2. Therefore Eq. (16b) may be written

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* The notion of this method has been introduced by Green[3], Stoker and Peters[12] in the treatment of some similar problems.

$$H(x, t) = -\frac{1}{\pi} R\ell \int_L \left\{ \frac{e^{ikx}}{\Lambda} - \frac{e^{i[kx+(\omega - Uk)t]}}{2\omega(\omega - Uk)} - \right.$$

$$\left. - \frac{e^{i[kx-(\omega + Uk)t]}}{2\omega(\omega + Uk)} \right\} kdk \tag{33}$$

where the contour L lies on the positive real axis except for two indentations, a semicircle in the lower half plane at κ_1 and another in the upper half plane at κ_2 (see Fig. 1). Since the exponent $i(\omega - Uk)t$ has its real part negative on both indentations, the second integral of (33) integrated on the semicircles makes a contribution that tends to zero(like t^{-1}) as $t \to +\infty$, <u>keeping x fixed</u>. The second integral on the remaining portions of L lying on the real axis is readily seen (e.g. by integration by parts, see Ref. 17, p. 46) to die out (like t^{-1}) as $t \to +\infty$ for fixed x. The last integral itself has no singularity on the real axis, so that the path L can be taken along the real axis. Thus, this integral can also be seen to die out (like t^{-2}) for fixed x. Therefore, as $t \to +\infty$, only the first integral of (33) remains,

$$H(x, t) \to -\frac{1}{\pi\sigma} R\ell \int_L \frac{e^{ikx}\, dk}{(k-\kappa_1)(k-\kappa_2)} . \tag{34}$$

The path L, once fixed by requiring that the transient terms integrated along L die out as $t \to +\infty$, may further be deformed as long as it does not pass across the singularities at κ_1 and κ_2. The steady state of the surface waves can then be calculated from the above integral by applying the theorem of residues to the integral on a closed contour Γ which consists of L and a large semicircle $|k| = R$ in the upper half plane for $x > 0$ (or in the lower half plane for $x < 0$) and back to the origin by the negative real axis. By passing to the limit $R \to \infty$, one obtains again Eq. (30).

This steady-state limit of the fundamental solution (Eq. (30)) shows that there are two trains of waves propagating respectively on each side of the origin. The wave on the downstream side, $\sin \kappa_1 x$, has wave length

$$\lambda_1 = \frac{2\pi}{\kappa_1} = \lambda_m \left[M^2 + (M^4 - 1)^{1/2} \right] > \lambda_m \quad \text{for} \quad M = \frac{U}{c_m} > 1$$

and is called the gravity wave; while the upstream wave has wave length $\lambda_2 = \lambda_m^2/\lambda_1$, less than λ_m, the so-called capillary wave. The last term in Eq. (30) represents the local effect which levels off rapidly from the origin. This integral can be reduced to known functions (see Ref. 1, p. 403):

$$L(x) = \int_0^\infty \frac{\cos kx\, dk}{(k+\kappa_1)(k+\kappa_2)} = \frac{1}{\kappa_2 - \kappa_1} \Big\{ \mathrm{Ci}\, \kappa_2 x \cos \kappa_2 x - \mathrm{Ci}\, \kappa_1 x \cos \kappa_1 x$$

$$\mathrm{si}\, \kappa_2 x \sin \kappa_2 x - \mathrm{si}\, \kappa_1 x \sin \kappa_1 x \Big\} \qquad (35a)$$

which Ci and si denote the cosine and sine integrals:

$$\mathrm{Ci}\, u = -\int_u^\infty \frac{\cos t}{t}\, dt, \qquad \mathrm{si}\, u = -\int_u^\infty \frac{\sin t}{t}\, dt. \qquad (35b)$$

From the known properties of these functions[15] one can easily verify that the integral L has the value $(\kappa_2 - \kappa_1)^{-1} \log(\kappa_2/\kappa_1)$ at $x=0$, becomes less than $0.04\, (\kappa_2 - \kappa_1)^{-1}$ for $x > \lambda_1/2$, and behaves like x^{-2} for $x \gg \lambda_1$. Since this last term of Eq. (30) only has a local effect, it has no mechanism for radiating energy away from the origin, as does the first term, and therefore contributes no wave resistance.

The natural consequence obtained above that the two waves occur separately on each side of the surface pressure is actually in accordance with the physical principle that governs the rate of energy transmission and is the solution sought for in the steady flow problem. Note that if the order of the limit and integration in $A_2(x)$ is reversed, integral A_2 then disappears. It is thus clear that in the direct formulation of the steady flow case (by dropping out all time derivatives), the result contains only A_1.

5. Asymptotic Solution for Large Time, $(U > c_m)$

A more detailed analysis is required to obtain the asymptotic solution of $H(x, t)$ for large, but finite, values of t. The integral representation (16b) is in a form suitable, at least in some sub-intervals of the integration, for the application of the method of stationary phase, (see Ref. 16, p. 515). As the exponents are purely imaginary for real k, the path of stationary phase is the real axis. Consequently it is no longer advantageous here to deform the path of integration. Also, when the first two terms of (16b) are integrated separately along the real k-axis, their Cauchy principal values are taken.

Write Eq. (16b) as

$$H(x, t) = H_1(x) + H_2(x, t) + H_3(x, t) \, , \tag{36a}$$

$$H_1 = -\frac{1}{\pi} P\int_0^{\infty} \frac{\cos kx}{\Lambda} \, kdk \, , \tag{36b}$$

$$H_2 = \int_0^{\kappa_1 - \epsilon} + P\int_{\kappa_1 - \epsilon}^{\kappa_1 + \epsilon} + \int_{\kappa_1 + \epsilon}^{\kappa_2 - \epsilon} + P\int_{\kappa_2 - \epsilon}^{\kappa_2 + \epsilon} + \int_{\kappa_2 + \epsilon}^{\infty} \frac{R\ell \exp i[k(x-Ut) + \omega t]}{2\pi\omega(\omega - Uk)} \, kdk \tag{36c}$$

$$= H_{21} + H_{22} + H_{23} + H_{24} + H_{25} \, , \qquad \text{according to the order,}$$

$$H_3 = \int_0^{\infty} \frac{R\ell \exp i[k(x - Ut) - \omega t]}{2\pi\omega(\omega + Uk)} \, kdk \, , \tag{36d}$$

where in (36c) ϵ is a small positive quantity, chosen to be much less than both κ_1 and $(\kappa_2 - \kappa_1)$. The above decomposition bears with it some physical significance. H_1 is independent of t and is even in x. H_2 as a whole represents the resultant dispersion of all the waves propagating upstream from the point $x = Ut$. Among these wave components H_{22} and H_{24} represent the interaction of the gravity wave $(k = \kappa_1)$ and the capillary wave ($k=\kappa_2$) with the motion of the external disturbance. Presumably $H_1+H_{22}+H_{24}$ will then describe the particular gravity wave and capillary wave that can accompany the external disturbance. As for the rest of H_2, H_{21} represents the dispersion of the gravity waves with $k<\kappa_1$, and H_{25}, the dispersion of the capillary waves with $k>\kappa_2$, whereas H_{23} represents the interaction of the gravity and capillary waves with $\kappa_1<k<\kappa_2$. Finally, H_3 represents the dispersion of the waves propagating downstream from $x=Ut$; it will then describe more or less the history of the downstream waves due to the initial disturbance.

Now $H_1(x)$ is identical to $A_1(x)$ of Eq. (24), hence

$$H_1(x) = -\frac{\operatorname{sgn} x}{\sigma(\kappa_2-\kappa_1)}\left\{\sin\kappa_1 x - \sin\kappa_2 x\right\} + \frac{1}{\pi\sigma}\, L(x) \tag{37}$$

with $L(x)$ given by Eq. (35).

In the evaluation of H_2 we shall restrict ourselves for simplicity to moderate and large values of $U>c_o$, so that $\kappa_1<\kappa_o<\kappa_m<\kappa_2$ (see Fig. 3). In H_{22} we use the expansion (31), taking z to be real, then

$$H_{22} \cong -\frac{R\ell}{2\pi a U}\, P\int_{-\epsilon}^{\epsilon}\left(1+a_1 z+0(z^2)\right) e^{i\left[\kappa_1 x+(x-at)z-\beta_1 t z^2+0(z^3)\right]}\,\frac{dz}{z}$$

in which a and β_1 are given by Eq. (31b), with $\beta_1>0$ here for $\kappa_1<\kappa_o$, and

$$a_1 = \frac{4\kappa_1^2+(\kappa_1+\kappa_2)^2}{4\kappa_1(\kappa_2^2-\kappa_1^2)} .$$

The order terms in the integrand above may be neglected since for t large, their contribution to the integral is of the order of $t^{-1-\nu}$ where $t^{-\nu}$ is the smallest order term retained. Then

$$H_{22} \cong -\frac{R\ell}{\pi\alpha U} e^{i\kappa_1 x}\left\{ i\int_0^{\epsilon\sqrt{\beta_1 t}} e^{-iz^2} \sin 2\gamma_1 z \frac{dz}{z} + \right.$$

$$\left. + \frac{a_1}{\sqrt{\beta_1 t}} \int_0^{\epsilon\sqrt{\beta_1 t}} e^{-iz^2} \cos 2\gamma_1 z \, dz \right\} \tag{38a}$$

with

$$\gamma_1 = (x - \alpha t)/(4\beta_1 t)^{1/2} . \tag{38b}$$

One may note that the upper limit $\epsilon\sqrt{\beta_1 t} \rightarrow \infty$ as $t \rightarrow \infty$, but γ_1 may be large (such as at $x = 0$) or small (such as at $x = \alpha t$). Now,

$$\int_0^{\infty} e^{-iz^2} \cos 2\gamma z \, dz = \frac{\sqrt{\pi}}{2} e^{i(\gamma^2 - \pi/4)} ;$$

$$\int_0^{\infty} e^{-iz^2} \sin 2\gamma z \frac{dz}{z} = \sqrt{\pi} \int_0^{\gamma} e^{i(u^2 - \pi/4)} du = \frac{\pi}{2}(1-i)\left[C\left(\sqrt{\frac{2}{\pi}}\gamma\right) + i\,S\left(\sqrt{\frac{2}{\pi}}\gamma\right)\right]$$

where C and S denote the Fresnel integrals:

$$C(u) = \int_0^u \cos\left(\frac{\pi t^2}{2}\right) dt , \qquad S(u) = \int_0^u \sin\left(\frac{\pi t^2}{2}\right) dt . \tag{39}$$

On the other hand, for t large

$$\int_{\epsilon\sqrt{\beta_1 t}}^{\infty} e^{-iz^2} \cos 2\gamma z \, dz = 0\left(\int_{\sqrt{t}}^{\infty} e^{-iz^2} dz\right) = 0\,(t^{-1/2})$$

upon integration by parts. Similarly,

$$\int_{\epsilon\sqrt{\beta_1 t}}^{\infty} e^{-iz^2} \sin 2\gamma z \frac{dz}{z} = 0\left(\int_{\sqrt{t}}^{\infty} e^{-iz^2} \frac{dz}{z}\right) = 0(t^{-1}) .$$

Therefore, after using these relations, Eq. (38a) becomes

$$H_{22} \cong \frac{1}{\sigma(\kappa_2-\kappa_1)}\left\{(C+S)\sin\kappa_1 x - (C-S)\cos\kappa_1 x - \right.$$

$$\left. - \frac{a_1}{\sqrt{\pi\beta_1 t}} \cos\left(\kappa_1 x + \gamma_1^2 - \frac{\pi}{4}\right) + O\left(\frac{1}{t}\right)\right\} \qquad (40)$$

in which the argument of C and S is of course $(\gamma_1\sqrt{2/\pi})$, γ_1 being given by Eq. (38b). The Fresnel integrals have the following properties: They are odd functions of the argument. For large values of u,

$$C(u) \cong \frac{1}{2}\,\mathrm{sgn}\,u + \frac{1}{\pi u}\sin\left(\frac{\pi}{2}u^2\right) + 0(u^{-2}),$$

$$S(u) \cong \frac{1}{2}\,\mathrm{sgn}\,u - \frac{1}{\pi u}\cos\left(\frac{\pi}{2}u^2\right) + 0(u^{-2});$$

and for small values of u,

$$C(u) \cong u + 0(u^5), \qquad\qquad S(u) \cong \frac{\pi}{6}u^3 + 0(u^7).$$

Hence for t and γ_1 both large (i.e., in the region $|x-at| >> \sqrt{4\beta_1 t}$).

$$H_{22} \cong \frac{1}{\sigma(\kappa_2-\kappa_1)}\left\{\mathrm{sgn}(x-at)\sin\kappa_1 x - \frac{1}{\sqrt{\pi}}\left(\frac{1}{\gamma_1} + \frac{a_1}{\sqrt{\beta_1 t}}\right)\cos\left(\kappa_1 x + \gamma_1^2 - \frac{\pi}{4}\right) + \right.$$

$$\left. + 0(\gamma_1^{-2}, t^{-1})\right\} \quad ; \qquad (41a)$$

and for t large but γ_1 small (or in the region $|x-at| << \sqrt{4\beta_1 t}$),

$$H_{22} \cong \frac{1}{\sigma(\kappa_2-\kappa_1)}\left\{\frac{2\gamma_1}{\sqrt{\pi}}\sin\left(\kappa_1 x - \frac{\pi}{4}\right) - \frac{a_1}{\sqrt{\pi\beta_1 t}}\cos\left(\kappa_1 x - \frac{\pi}{4}\right) + 0(\gamma_1^3, t^{-1})\right. .$$

(41b)

It is of interest to note that the first term in Eq. (41a) changes sign at $x = \alpha t (>0)$, thereby canceling the gravity wave in H_1 (see Eq. (37)) for $x \leq 0$ and for $x \gg \alpha t + \sqrt{4\beta_1 t}$, but doubling it in $0 \leq x \ll \alpha t - \sqrt{4\beta_1 t}$.

In the integral H_{24}, we use the expansion (32) and proceed in a similar way as above; then we obtain for t large,

$$H_{24} \cong \frac{-1}{\sigma(\kappa_2 - \kappa_1)} \left\{ (C+S)\sin\kappa_2 x + (C-S)\cos\kappa_2 x + \right.$$
$$\left. + \frac{a_2}{\sqrt{\pi\beta_2 t}} \cos(\kappa_2 x - \gamma_2^2 + \tfrac{\pi}{4}) + O(\tfrac{1}{t}) \right\} \tag{42a}$$

where

$$\gamma_2 = \frac{x + \alpha t}{\sqrt{4\beta_2 t}}, \qquad a_2 = \frac{4\kappa_2^2 + (\kappa_1 + \kappa_2)^2}{4\kappa_2(\kappa_2^2 - \kappa_1^2)} \tag{42b}$$

and C, S here stand for the Fresnel integrals with argument $(\gamma_2 \sqrt{2/\pi})$. Thus for t and γ_2 both large (in the region $|x + \alpha t| \gg \sqrt{4\beta_2 t}$),

$$H_{24} \cong \frac{-1}{\sigma(\kappa_2 - \kappa_1)} \left\{ \operatorname{sgn}(x + \alpha t)\sin\kappa_2 x + \frac{1}{\sqrt{\pi}}\left(\frac{1}{\gamma_2} + \frac{a_2}{\sqrt{\beta_2 t}}\right)\cos(\kappa_2 x - \gamma_2^2 + \tfrac{\pi}{4}) + \right.$$
$$\left. + 0(\gamma_2^{-2}, t^{-1}) \right\}; \tag{43a}$$

and for t large, but γ_2 small (in the region $|x + \alpha t| \ll \sqrt{4\beta_2 t}$)

$$H_{24} \cong \frac{-1}{\sigma(\kappa_2 - \kappa_1)} \left\{ \frac{2\gamma_2}{\sqrt{\pi}} \sin(\kappa_2 x + \tfrac{\pi}{4}) + \frac{a_2}{\sqrt{\pi\beta_2 t}} \cos(\kappa_2 x + \tfrac{\pi}{4}) + 0(\gamma_2^3, t^{-1}) \right\}. \tag{43b}$$

Note also that the first term in Eq. (43a) changes sign at $x = -\alpha t (<0)$ and hence cancels the capillary wave in H_1 for $x \geq 0$ and $x \ll -\alpha t - \sqrt{4\beta_2 t}$, but doubles it in $0 \geq x \gg -\alpha t + \sqrt{4\beta_2 t}$.

The integral H_{21} represents the dispersion of all gravity waves having wave lengths greater than λ_1. Since the capillary effect on the wave velocity of these gravity waves is rather small (the effect is less than 5% here for $k < \kappa_1 < \kappa_o = 0.393\ \kappa_m$, as can be verified from Eq. (19)), the surface tension may be neglected in H_{21}. Then from Eq. (15),

$$\omega = (g\,k)^{1/2}, \qquad \omega' = \frac{1}{2}(g/k)^{1/2}, \qquad \omega'' = -\frac{1}{4}(g/k^3)^{1/2}. \tag{44}$$

The method of stationary phase may readily be applied to calculate H_{21} for large values of t. Write the exponential function in (36c) as $\exp\left\{i\,t\,f(k)\right\}$ with

$$f(k) = \omega(k) - \xi\,k, \qquad \xi = U - (x/t). \tag{45}$$

Then the stationary point k_o is given by $f'(k_o) = 0$, or

$$\omega'(k_o) = \xi, \quad \text{and from (44),} \quad k_o = g(2\xi)^{-2}. \tag{46}$$

Since ω' is positive, k_o may exist only for positive ξ. Furthermore, k_o must also fall inside the range of integration of H_{21}, $0 < k_o \lesssim \kappa_1 - \epsilon$. Hence, with $\kappa_1 = g/U^2$ when σ is neglected, we must have

$$\xi > \frac{U}{2}\left(1 + \frac{U^2\epsilon}{2g}\right), \qquad \text{or} \qquad x < \frac{Ut}{2}\left(1 - \frac{U^2\epsilon}{2g}\right).$$

For a given pair (x, t) satisfying this inequality, there corresponds a stationary point k_o, given by (46), at which

$$\omega(k_o) = g/2\xi, \qquad \omega''(k_o) = -2\xi^3/g, \qquad f(k_o) = g/(4\xi).$$

Hence following the principle of stationary phase, we may expand the integrand about k_o to yield

$$H_{21} \cong \frac{1}{2\pi}\left[\frac{k}{\omega(\omega - Uk)}\right]_{k=k_o} R\ell\ e^{itf(k_o)} \int_{-\infty}^{\infty} e^{\frac{1}{2} it\omega''(k_o)\epsilon^2}\, d\epsilon + O\!\left(\frac{1}{t}\right)$$

$$\cong \left[\pi g(Ut - x)\right]^{-1/2} \frac{t}{Ut - 2x} \cos\left[\frac{g\,t^2}{4(Ut - x)} - \frac{\pi}{4}\right] + O\!\left(\frac{1}{t}\right), \tag{47a}$$

valid for t large and $x \ll Ut/2$. In the region $Ut/2 - \sqrt{U^3 t/g} < x < Ut/2$, the higher term in the expansion of $k/[\omega(\omega - Uk)]$ should be included to give the final result:

$$H_{21} \cong \frac{1}{U^2}\left(\frac{2g}{\pi Ut}\right)^{1/2}\left(\frac{Ut-2x}{U}\right)\sin\left(\frac{gx}{U^2}-\frac{\pi}{4}\right)+O\left(\frac{1}{t}\right). \qquad (47b)$$

For $x \geq Ut/2$ the integrand has no stationary point, and therefore $H_{21} = 0(1/t)$ in that region. For fixed large values of t and $x \ll Ut/2$, the wave represented by (47a) decreases in amplitude towards the upstream; the wave length of the wave passing x at t is

$$\lambda = \frac{8\pi}{g}(U - x/t)^2$$

which decreases more and more, the closer the wave is to the point $x = Ut/2$. At a fixed point $x \ll Ut/2$, this wave decays like $t^{-1/2}$.

Next, consider the capillary wave represented by H_{25} in Eq. (36c). By a similar argument the gravity g may be neglected in H_{25} so that $\omega = (\sigma k^3)^{1/2}$. Application of the method of stationary phase yields the asymptotic representation for large values of t:

$$H_{25} \cong \frac{27}{4}\left(\frac{\sigma}{\pi t}\right)^{1/2}\left(\frac{t}{Ut-x}\right)^{5/2}\left(\frac{-t}{2x+Ut}\right)\cos\left[\frac{4}{27}\frac{(Ut-x)^3}{\sigma t^2}-\frac{\pi}{4}\right]+O\left(\frac{1}{t}\right) \qquad (48a)$$

for $x \ll -Ut/2$;

$$\cong -\frac{1}{U^2}\left(\frac{2U}{3\pi\sigma t}\right)^{1/2}(2x+Ut)\sin\left(\frac{U^2 x}{\sigma}+\frac{\pi}{4}\right)+O\left(\frac{1}{t}\right) \qquad (48b)$$

for $-Ut/2 - \sqrt{\sigma t/U} < x < -Ut/2$,

and $H_{25} = 0(1/t)$ for $x \geq -Ut/2$. This wave has the local wave length

$$\lambda = \frac{9}{2}\pi\sigma\left(\frac{t}{Ut-x}\right)^2 ,$$

which decreases more and more with decreasing amplitude, the farther upstream the wave is from the point $x = -Ut/2$. Its amplitude also decays like $t^{-1/2}$ at any point $x \ll -Ut/2$.

In the integral H_{23} the significant features of the interaction between the capillary and gravity waves can be acquired by evaluating the integral in a neighborhood of κ_o where ω'' vanishes (see Eq. (27)). Thus, with $k = \kappa_o + \epsilon$, one obtains for $|\epsilon| \ll \kappa_o$,

$$\omega(k) = \omega_o + c_{g_o}\epsilon + \frac{1}{6} c_{g_o}'' \epsilon^3 \tag{49a}$$

where

$$\kappa_o = 0.393\ \kappa_m, \qquad \omega_o = \omega(\kappa_o) = 0.477\ \omega_m, \qquad c_o = \omega_o/\kappa_o = 1.212\ c_m$$

$$c_{g_o} = \omega'(\kappa_o) = 0.767\ c_m, \qquad c_{g_o}'' = \omega'''(\kappa_o) = 1.484\ c_m/\kappa_m^2 . \tag{49b}$$

By using this approximation, we have for $U > c_o$, and t large*,

$$H_{23} \cong \frac{R\ell}{2\pi\omega_o(c_o - U)} e^{i\kappa_o t(c_o - \xi)} \int_{-(\kappa_o - \kappa_1 - \epsilon)}^{(\kappa_2 - \kappa_o - \epsilon)} e^{it\left[(c_{g_o} - \xi)\epsilon + \frac{1}{6} c_{g_o}'' \epsilon^3\right]} d\epsilon$$

$$\cong \left(\frac{2}{t c_{g_o}''}\right)^{1/3} \frac{\cos\kappa_o\left[x - (U - c_o)t\right]}{\kappa_o c_o (c_o - U)} \mathrm{Ai}\left\{\left(\frac{2}{t c_{g_o}''}\right)^{1/3}\left[x - (U - c_{g_o})t\right]\right\} + 0(t^{-2/3}) \tag{50}$$

where Ai denotes the Airy integral:

$$\mathrm{Ai}(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos\left[\frac{v^3}{3} + z v\right] dv . \tag{51}$$

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* It should be noted that when $c_m < U \leq c_o$, the integrals H_{22} and H_{23} coincide in range of integration and thus reduce to one integral. In such a case the denominator of the integrand in Eq. (50) should include the terms up to $0(\epsilon)$; the final result then contains the integral of the Airy function.

The asymptotic behavior of $Ai(z)$ for large or small values of z are given elsewhere (e.g. see Ref. 16, p. 508 and Ref. 17); it decreases exponentially for z positive and large, and oscillates with decreasing amplitude for negative large z. For large t, the function Ai in Eq. (50) varies much more slowly with x than the cosine wave because of the small factor $t^{-1/3}$ in its argument. Hence the profile H_{23} behaves like a train of cosine waves of length $\lambda_o = 2\pi/\kappa_o$, propagating downstream at velocity $(U - c_o)$, but its amplitude falls off exponentially for $x > (U - c_{go})t$ and oscillates slowly with decreasing amplitude for $x < (U - c_{go})t$. Near the point $x = Ut$, the amplitude decays like $t^{-1/3}$, a rate slower than $t^{-1/2}$, the rate of decay of the rest of the transient waves.

The integral H_3 of Eq. (36) represents the resultant dispersion of the waves going downstream; its asymptotic value for large t is

$$H_3(x, t) \cong \left(\frac{2}{t c_{go}\pi}\right)^{1/3} \frac{\cos \kappa_o[x - (U + c_o)t]}{\kappa_o c_o (U + c_o)} \ \mathrm{Ai}\left\{\left(\frac{2}{t c_{go}\pi}\right)^{1/3} \left[(U + c_{go})t - x\right]\right\} +$$

$$+ \, 0(t^{-2/3}) \qquad \text{for} \quad x \text{ near } Ut . \tag{52}$$

For $x \geq 3/2\ Ut$. H_3 is asymptotic to the expression H_{21} of Eq. (48) with $(Ut-x)$ and $(Ut-2x)$ replaced by their absolute values; and for $x \geq 5/2\ Ut$, $H_3 \cong -H_{25}$ of Eq. (49) again with $(Ut-x)$ and $(2x+Ut)$ replaced by $(x-Ut)$ and $|2x+Ut|$; and for $x \ll Ut$, $H_3 = 0(1/t)$.

Finally, in order to summarize the detailed information above and thereby to exhibit a clear over-all picture, we consider for simplicity the case $U \gg c_m$ so that

$$\begin{aligned} &\kappa_1 \cong g/U^2, \quad \kappa_2 \cong U^2/\sigma, \quad \sigma(\kappa_2 - \kappa_1) \cong U^2, \quad \alpha \cong U/2 \\ &\beta_1 \cong U^3/8g, \quad \beta_2 \cong \frac{3\sigma}{8U}, \quad a_1 \cong U^2/4g, \quad a_2 \cong 5\sigma/4U^2 , \end{aligned} \tag{53}$$

and hence $\kappa_2 \gg \kappa_1$, $\beta_1 \gg \beta_2$, $a_1 \gg a_2$. The resultant asymptotic representation of $H (= H_1 + H_2 + H_3)$ for t large is given for different regions in space as follows, the order of error term being omitted as understood. For

$$-\frac{1}{2}Ut < x < -\frac{1}{2}Ut + \left(\frac{\sigma t}{U}\right)^{1/2},$$

$$H \cong -\frac{1}{U^2}\left\{\sin\frac{U^2 x}{\sigma} + \left(\frac{2U}{3\pi\sigma t}\right)^{1/2}(2x+Ut)\sin\left(\frac{U^2 x}{\sigma}+\frac{\pi}{4}\right)\right\}; \tag{54a}$$

for

$$-\frac{1}{2}Ut + \left(\frac{\sigma t}{U}\right)^{1/2} \ll x \le 0,$$

$$H \cong -\frac{1}{U^2}\left\{2\sin\frac{U^2 x}{\sigma} - \frac{U^2}{\pi\sigma}L(x) + M(x,t)\right. \tag{54b}$$

where $L(x)$ is given by Eq. (35) and

$$M(x,t) = \frac{5}{U}\left(\frac{2\sigma}{3\pi Ut}\right)^{1/2}\left(\frac{x+Ut}{2x+Ut}\right)\cos\left[\frac{4U}{3\sigma t}\left(x^2 - \frac{xUt}{2} + \frac{U^2t^2}{4}\right) - \frac{\pi}{4}\right],$$

for

$$0 \le x \ll +\frac{1}{2}Ut - \left(\frac{U^3 t}{2g}\right)^{1/2},$$

$$H \cong -\frac{1}{U^2}\left\{2\sin\frac{gx}{U^2} - \frac{U^2}{\pi\sigma}L(x) + \left[\frac{U^2}{\pi g(Ut-x)}\right]^{1/2}\left(\frac{Ut}{Ut-2x}\right)\cos\left[\frac{gt^2}{4(Ut-x)} - \frac{\pi}{4}\right] + N\right\}, \tag{54c}$$

$$N(x,t) = \left(\frac{U}{8\pi gt}\right)^{1/2}\left(\frac{2x+Ut}{2x-Ut}\right)\cos\left[\frac{2g}{U^3 t}\left(x^2 - \frac{xUt}{2} + \frac{U^2t^2}{4}\right) - \frac{\pi}{4}\right];$$

for

$$\frac{1}{2}Ut - \left(\frac{U^3 t}{g}\right)^{1/2} < x < \frac{1}{2}Ut,$$

$$H \cong -\frac{1}{U^2}\left\{\sin\frac{gx}{U^2} + \left(\frac{8g}{\pi U^3 t}\right)^{1/2}(Ut-2x)\sin\left(\frac{gx}{U^2} - \frac{\pi}{4}\right)\right\}, \tag{54d}$$

for

$$\frac{1}{2}\,Ut < x < \frac{1}{2}\,Ut + \left(\frac{U^3 t}{g}\right)^{1/2} ,$$

$$H \cong -\frac{1}{U^2}\left\{\sin\frac{gx}{U^2} + \left(\frac{2g}{\pi U^3 t}\right)^{1/2}(Ut - 2x)\,\sin\left(\frac{gx}{U^2} - \frac{\pi}{4}\right)\right\} , \quad (54e)$$

for

$$\frac{1}{2}\,Ut + \left(\frac{U^3 t}{2g}\right)^{1/2} \ll x < \frac{3}{2}\,Ut$$

$$H \cong -\frac{1}{U^2}\left(\frac{16}{t\,cg_o''}\right)^{1/3}\frac{\cos\kappa_o(x - Ut)}{\kappa_o c_o}\left\{Ai\left[\left(\frac{2}{t\,cg_o''}\right)^{1/3}(x - Ut)\right] - \right.$$

$$\left. - Ai\left[\left(\frac{2}{t\,cg_o''}\right)^{1/3}(Ut - x)\right]\right\} - \frac{N(x,t)}{U^2} \quad (54f)$$

The asymptotic representation of H for $x \ll -\frac{1}{2}\,Ut$ and $x > \frac{3}{2}\,Ut$ are omitted here because their magnitudes are rather unimportant. Based on this asymptotic solution a qualitative picture of the wave form is drawn in Fig. 4.

This result shows that on the upstream side, a train of capillary waves of wave length $\lambda_2 = 2\pi\sigma/U^2$ exists in $0 \geq x > -Ut/2$; this wave falls off in amplitude beyond $x = -Ut/2$ with decreasing wave length. A train of gravity wave of length $\lambda_1 = 2\pi U^2/g$ propagates on the downstream side in $0 \leq x < Ut/2$, tapering off beyond $x = Ut/2$. The phenomenon of the interaction between the initially generated gravity and capillary waves takes place near $x = Ut$; the net effect produces near $x = Ut$ a train of waves of length $\lambda_o = 2\pi/\kappa_o$ (equal to 4.6 cm for a water surface), propagating downstream at velocity U, oscillating slowly with frequency $\omega_o/2\pi$ and dying out at a rate proportional to $t^{-1/3}$. The rest of the transient waves that depend on t dies out like $t^{-1/2}$. It may also be remarked that this asymptotic solution is consistent with the physical principle of energy transmission that in the present coordinate system the energy of the gravity wave is transmitted only downstream at the rate $U/2$ and the energy of the capillary wave, only upstream at the same rate.

In the present case of $U \gg c_m$ the characteristic time of importance is $\tau_1 = U/g$, the other characteristic time associated with surface tension, $\tau_2 = \sigma U^{-3}$, being much less than τ_1 for $U \gg c_m$. In terms of τ_1 the spatial range for Eq. (54c, d) can be written as

$$\frac{1}{2} Ut - (U^3 t/g)^{1/2} = \frac{1}{2} Ut \left[1 - (2\tau_1/t)^{1/2} \right] ;$$

the last quantity $(2\tau_1/t)^{1/2}$ may be considered small for $t \geq 100\, \tau_1$. Therefore, the above asymptotic solution for large time becomes valid for $t > 100\, \tau_1$, that is, after about eight gravity waves $(Ut/2 > 8\lambda_1)$ have been established.

As $t \to \infty$, the region of the monochromatic gravity wave and capillary wave extends to infinity and all transient waves diminish. It is in this manner that the limiting solution, as obtained in the last section, is approached.

6. The Case $U < c_m$

When U is less than c_m, the two zeros of $\Lambda(k)$ are complex conjugate, say κ and $\bar{\kappa}$ where (see Eq. (18))

$$\kappa/\kappa_m = M^2 + i(1 - M^4)^{1/2} , \qquad M = U/c_m < 1 . \tag{55}$$

Regarding k to be complex, the integrand of H_1 has two simple poles at κ and $\bar{\kappa}$. If one chooses a closed contour bounded by the first or the fourth quadrant and applies the theorem of residues, one readily obtains

$$H_1(x) = -\frac{1}{\pi\sigma}\int_0^\infty \frac{\cos kx\,dk}{(k-\kappa)(k-\bar{\kappa})}$$

$$= -\frac{2\cos(M^2\kappa_m x)}{(c_m^4 - U^4)^{1/2}}\, e^{-\kappa_m|x|(1-M^4)^{1/2}} + \frac{4U^2}{\pi c_m^4}\int_0^\infty \frac{e^{-\kappa_m|x|u}\, u\,du}{(u^2-1)^2+4M^4u^2}\,. \tag{56}$$

The first term represents a wave of length $2\pi/(\kappa_m M^2)$, but its amplitude falls off exponentially away from the origin. The second term, even in x, may be shown to damp out like x^{-2} for large x (a straightforward result by applying Watson's lemma, e.g., see Ref. 16, p. 501). In a similar way one can show that the transient waves H_2 and H_3 die out like t^{-2} as $t \to \infty$, so that the limiting solution of H has the same value as H_1 given above. Thus, in this case no surface wave (in the usual sense) can be generated.

In particular, as $U \to 0$, the second term with the integral in Eq. (56) tends numerically to one-half of the first term. Hence

$$H(x) = -\frac{1}{2\sqrt{g\sigma}}\, e^{-(g/\sigma)^{1/2}|x|} \qquad \text{at} \qquad U = 0 \tag{57}$$

which is the fundamental solution of $gH - \sigma H_{xx} = -\delta(x)$, the corresponding limit of Eq. (4).

7. The Case $U = c_m$

As $U \to c_m$, $\kappa_1 \to \kappa_2 \to \kappa_m$, both Eq. (30) and (56) become meaningless. This result would imply that no steady solution really exists when $U = c_m$. To see this, we put $\Lambda = \sigma k(k-\kappa_m)^2$ in Eq. (12) and further introduce the nondimensional quantities:

$$u = k/\kappa_m, \qquad \xi = \kappa_m x, \qquad \tau = \omega_m t, \qquad z = s/\omega_m \qquad (\omega_m = c_m\kappa_m);$$

then

$$H = -\frac{1}{c_m^2 \pi i} \int_\Gamma e^{\tau z} \frac{dz}{z} \int_0^\infty \frac{\left[(u-1)^2 + 2z^2/u\right] \cos \xi u + 4z \sin \xi u}{\left[(u-1)^2 + 2z^2/u\right]^2 + (4z)^2} du . \tag{58}$$

To obtain the asymptotic representation of H for large τ, we neglect higher powers of z other than the terms necessary to keep the integral convergent. Thus, for t large,

$$H(x,t) \cong -\frac{1}{c_m^2 \pi^2 i} \int_\Gamma e^{\tau z} \frac{dz}{z} \int_0^\infty \frac{(u-1)^2 \cos \xi u + 4z \sin \xi u}{(u-1)^4 + 4z^2 (u+1)^2/u} du$$

$$\cong -\frac{4}{\pi c_m^2} \int_0^\infty \left\{ \frac{\cos \xi u}{(u-1)^2} \sin^2 \left[\frac{\sqrt{u}(u-1)^2 \tau}{4(u+1)} \right] + \frac{\sqrt{u} \sin \xi u}{(u+1)(u-1)^2} \sin \left[\frac{\sqrt{u}(u-1)^2 \tau}{2(u+1)} \right] \right\} du .$$

It can be seen that for τ large, the most significant contribution to the integral comes from a neighborhood of $u = 1$. Thus, one may divide the interval into two parts: from $u = 0$ to 2 and from 2 to ∞; the integral over the second interval can be estimated to be of the order τ^{-1} upon integration by parts. For the integral in the interval $0 \leq u \leq 2$, one may expand the functions about the point $u = 1$. Hence, with $v = u - 1$,

$$H(x,t) \cong -\frac{8}{\pi c_m^2} \left\{ \cos \xi \int_0^1 \sin^2 \left(\frac{\tau v^2}{8} \right) \frac{dv}{v^2} + \frac{1}{2} \sin \xi \int_0^1 \sin \left(\frac{\tau v^2}{4} \right) \frac{dv}{v^2} \right.$$

$$\cong -\frac{(2\tau)^{1/2}}{\pi c_m^2} \left\{ \cos \xi \int_0^\infty \frac{\sin^2 u}{u^{3/2}} du + \frac{1}{\sqrt{2}} \sin \xi \int_0^\infty \frac{\sin u}{u^{3/2}} du \right\} + 0(t^{-1/2})$$

$$\cong -\frac{2}{c_m^2} \left(\frac{\omega_m t}{\pi} \right)^{1/2} \cos \left(\kappa_m x - \frac{\pi}{4} \right) + 0(t^{-1/2}) , \tag{59}$$

which grows beyond all bounds as $t \to \infty$. In order to explain this result, one must recall the assumptions and conditions under which this solution is valid. In the linearization, the point of application of the surface pressure is approximated to be on $y = 0$. When the pressure moving at velocity c_m is exerted strictly on the deflected free surface, the wave would probably still grow until its amplitude becomes so large that the linear theory breaks down.

8. The Effect of Superposition

In the foregoing, $H(x, t)$ is actually the fundamental solution when the surface force p_o acts like the Dirac function. If p_o is distributed over an area on the surface, superposition of $H(x, t)$ leads in general to the suppression of the wave amplitude, a quite well-known result. Following Lamb (see Ref. 1, p. 467) we consider the surface pressure

$$p_o(x) = \frac{\rho}{\pi} \cdot \frac{b}{b^2 + x^2} \qquad (b > 0) \tag{60}$$

which spreads more, the larger the value of b. Note also that this function tends to $\rho\delta(x)$ as $b \to 0$. After substituting this $p_o(x)$ and Eq. (12) into (11), one finds that $\zeta(x, t)$ has the same expression as $H(x, t)$ of Eq. (12) except for an additional factor $\exp(-kb)$ in the integrand. The corresponding steady-state solution is then

$$\zeta(x) = -\frac{2}{\sigma(\kappa_2 - \kappa_1)} \left\{ \begin{matrix} e^{-b\kappa_1} \sin \kappa_1 x \\ \\ e^{-b\kappa_2} \sin \kappa_2 x \end{matrix} \right\} \quad \text{for} \quad \begin{matrix} x \geq 0 \\ \\ x \leq 0 \end{matrix} \tag{61}$$

where the local deflection near the origin has been omitted. The exponential factors show the attenuation due to superposition, or, the effect of "interference"; the attenuation is more marked for the capillary waves, since $\kappa_2 > \kappa_1$.

In the case of $U < c_m$, one can obtain in a similar way an attenuation factor $\exp(-bU^2/2\sigma)$ to the first term of Eq. (56) in the expression for $\zeta(x, t)$: while in the case $U = c_m$, this factor becomes $\exp(-\kappa_m b)$.

9. The Viscous Effect

The damping of a train of simple wave due to the viscous effect has been treated by using a linear theory (Ref. 1, p. 625). One of the essential results may be cited here: If at time $t = 0$ a train of simple waves is established and is given as $\zeta = \zeta_o \sin k(x-Ut)$ where the liquid medium is otherwise at rest, and for $t > 0$ this wave is allowed to decay under the sole influence of viscosity, then the result is

$$\zeta(x, t) = \zeta_o \exp(-2\nu k^2 t) \sin k(x-Ut)$$

where ν is the kinematic viscosity of the liquid. It is of interest to note that the viscous effect only attenuates the wave amplitude without affecting the wave length, its velocity or its phase.

In the present case, the viscous damping of the surface waves generated and maintained by a surface pressure is, of course, more complicated. Some rough estimate of the viscous effect on this type of motion, however, can readily be carried out as follows. Our previous result indicates that the region of the simple wave extends outwards from the origin at a rate equal to $U/2$. Physically, this would mean the following. In a coordinate system fixed in space, as the force passes over a certain point at some instant, a wavelet is being created. After the force passes by, this wavelet is left behind to propagate as a free wave and to decay in the absence of maintaining force, hence its time rate of decay should obey the above law. But at the end of a large time interval t, this wavelet, after the net influence of dispersion, is at a distance $x = (1/2)Ut$ from the moving force. It is then a reasonable conjecture that the above result can be applied to the present case by replacing t in the exponent by $2x/U$. Thus, back to the coordinate system fixed with the force, the steady solution under the viscous influence should be approximately

$$H(x) = -\frac{2}{\sigma(\kappa_2-\kappa_1)}\left\{\begin{array}{c}\exp(-4\nu\kappa_1^2 x/U)\sin\kappa_1 x\\ \\ \exp(-4\nu\kappa_2^2 x/U)\sin\kappa_2 x\end{array}\right\} \text{ for } \left\{\begin{array}{l}x\geq 0\ ,\\ \\ x\leq 0\ .\end{array}\right. \tag{62}$$

Let us define an attenuation length

$$\ell = \frac{U}{4\nu k^2} = \left(\frac{c_m}{4\nu\kappa_m^2}\right)\left(\frac{\lambda}{\lambda_m}\right)^2\left[\frac{1}{2}\left(\frac{\lambda}{\lambda_m}+\frac{\lambda_m}{\lambda}\right)\right]^{1/2}$$

which corresponds to an attenuation factor e^{-1} at $x=\ell$ for the wave of length λ. For a water surface, $(c_m/4\nu\kappa_m^2) = 44$ cm. Hence when $\lambda=\lambda_m = 1.73$ cm, $\ell_m = 44$ cm. For longer waves, ℓ increases very rapidly since it is then proportional to $(\lambda/\lambda_m)^{5/2}$. For example, when $\lambda_1 = 173$ cm, then $\ell_1 = 3$ kilometers. Therefore, the viscous damping is effective only on the short waves.

By combining the effects of superposition and viscosity, we finally obtain a picture which would come very close to the physical observation.

10. Wave Resistance

Inasmuch as the surface pressure exerts itself always normal to the free surface when viscosity is omitted, the wave drag R must be equal to the horizontal component of the total surface force in the upstream direction, that is

$$R = -\int_{-\infty}^{\infty}\frac{\partial\zeta}{\partial x}\,p_o(x)\,dx\ . \tag{63}$$

We consider first the fundamental case: $p_o = \rho B\delta(x)$, $\zeta = BH(x,t)$ with B being a constant, then from Eq. (16),

$$R = -B^2 \left(\frac{\partial H}{\partial x}\right)_{x=0} = \frac{B^2}{\pi} \int_0^\infty \frac{k^2}{2\omega} \left\{ \frac{\sin(\omega - Uk)t}{\omega - Uk} - \frac{\sin(\omega + Uk)t}{\omega + Uk} \right\} dk \, . \tag{64}$$

For $U > c_m$ and t large, it follows from Riemann-Lebesgue lemma that the asymptotic value of R comes from the first integral in the interval near $k = \kappa_1$ and κ_2. Using the substitutions (31) and (32), one can verify that

$$R \cong \frac{B^2(\kappa_1 + \kappa_2)}{2\pi a U} \int_{-\epsilon_1}^{\epsilon_1} \frac{\sin a\epsilon t}{\epsilon} \, d\epsilon \rightarrow \frac{B^2(\kappa_1 + \kappa_2)}{\sigma(\kappa_2 - \kappa_1)} \, , \quad \text{as} \quad t \rightarrow \infty \, . \tag{65}$$

With the additional effect of superposition, we take $p_o(x) = (\rho B/\pi b)(1 + x^2/b^2)^{-1}$, then it can be shown that

$$R \cong \frac{B^2}{\sigma(\kappa_2 - \kappa_1)} \left[\kappa_1 e^{-2b\kappa_1} + \kappa_2 e^{-2b\kappa_2} \right] \quad \text{as} \quad t \rightarrow \infty \, . \tag{66}$$

When U is less than c_m, R vanishes because in this case p_o is even while $\partial\zeta/\partial x$ is odd in x and vice versa.

Discussion

Physically speaking, a liquid, such as water, is a highly dispersive medium for surface waves, and is, of course, also a viscous medium which resists, more or less, any flow motion. The fact that the present formulation based on the assumption of inviscid fluid yields automatically the correct solution implies that the important physical mechanism causing the resultant wave form must be dispersion not the viscosity, because the dispersive effect is already included in the boundary conditions of the problem. The viscous damping is effective only for the short capillary waves; the long gravity waves are hardly affected at all. Therefore, any remedy sought in the steady-state problem to obtain the correct solution by resorting to viscous damping is physically irrelevant.

Next, we can identify Rayleigh's frictional coefficient μ as approximately corresponding to the Laplace variable s used in this paper. The Laplace transform of Eq. (4) is

$$\left(U \frac{\partial}{\partial x} + s\right)\bar{\varphi} + \left(g - \sigma \frac{\partial^2}{\partial x^2}\right)\bar{\zeta} = -\frac{1}{\rho s} p_o(x) ;$$

while the corresponding boundary condition for the steady flow using Rayleigh's coefficient μ is

$$\left(U \frac{\partial}{\partial x} + \mu\right)\varphi + \left(g - \sigma \frac{\partial^2}{\partial x^2}\right)\zeta = -\frac{1}{\rho} p_o(x) .$$

From this pair of equations one immediately sees the resemblance of μ with s. But since s also appears in the transform of Eq. (3) to which μ is not introduced according to Rayleigh, s and μ are not identical. The process of taking the limit of $s \to 0$ in order to obtain the limiting solution therefore corresponds to Rayleigh's argument of letting $\mu \to 0$. The resemblance of μ and s may also be seen by comparing Eq. (12) with Rayleigh's solution (Ref. 1, p. 465), put partly in our notation,

$$\zeta = -\frac{1}{\pi}\int_0^\infty \frac{\Lambda \cos kx + \mu Uk \sin kx}{\Lambda^2 + (\mu Uk)^2}\, kdk .$$

This comparison reveals that $\mu \cong 2s$ as they both tend to zero. Even more interesting is the case $U = c_m$, in which case Rayleigh's solution (Ref. 1, p. 468) reads:

$$\zeta = -\frac{1}{(\sigma c_m \mu)^{1/2}} \cos\left(\kappa_m x - \frac{\pi}{4}\right) .$$

Lamb's explanation for this result (Ref. 1, p. 467) is: "to get an intelligible result in this case it is necessary to retain the frictional coefficient μ". However, if one regards this μ to be $(2s)$ and thereby performs the Laplace inversion together with the factor s^{-1}

in Eq. (12), one readily obtains Eq. (59). Therefore, we may assert that Rayleigh's force which is itself an ingenious device, is actually a time-limiting factor.

Mathematically speaking, the determination of the boundary condition at infinite distance in a steady-state problem involves two limiting processes, one for the space, the other for the time. It is found here that the exchange of the order of these limits is not justified. It is hoped that these points may serve their purpose to clarify the difficulties that may arise in some similar problems.

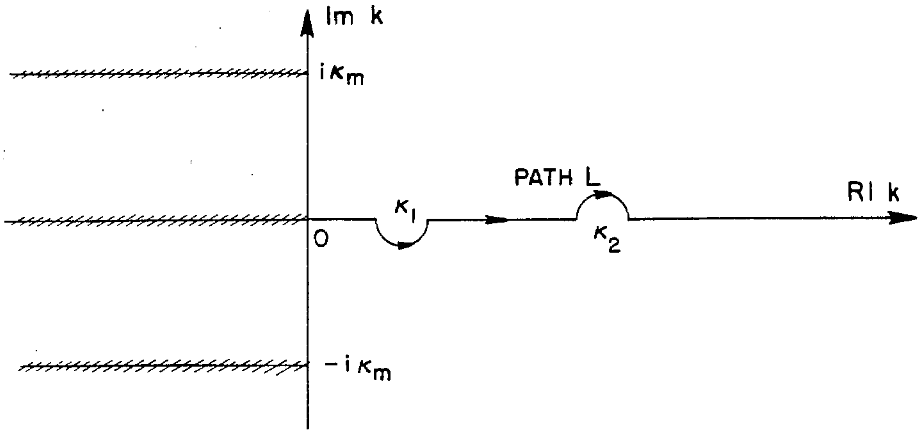
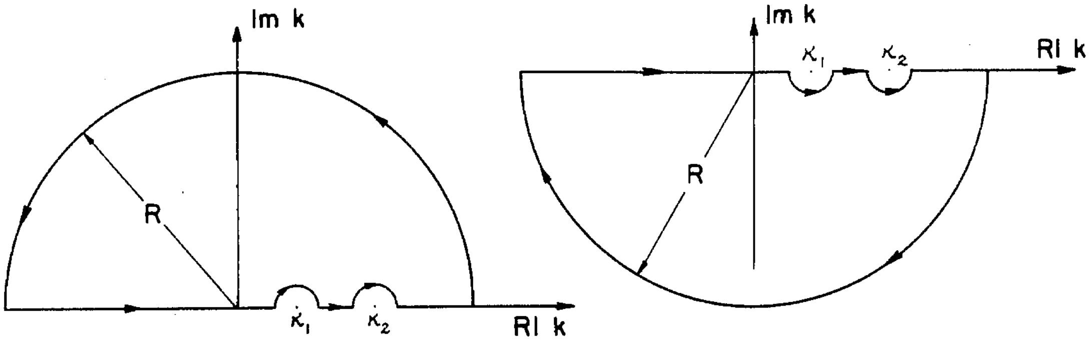


Fig. 1



THE PATH FOR $X>0$ THE PATH FOR $X<0$

Fig. 2

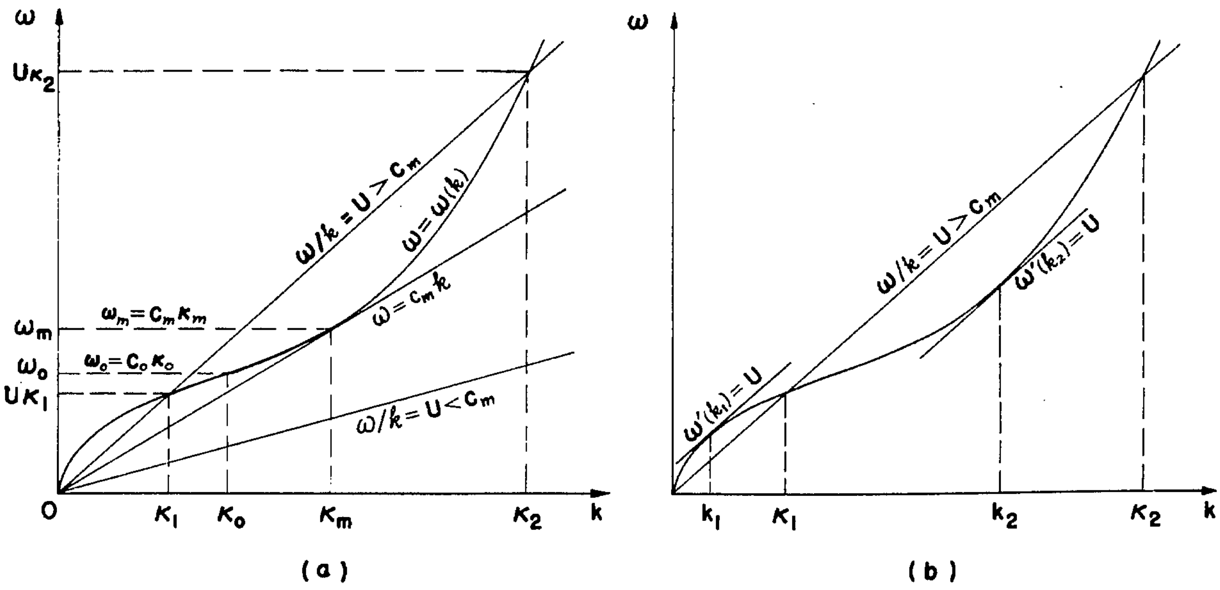


Fig. 3. Qualitative features of the curve $\omega = \omega(k)$.

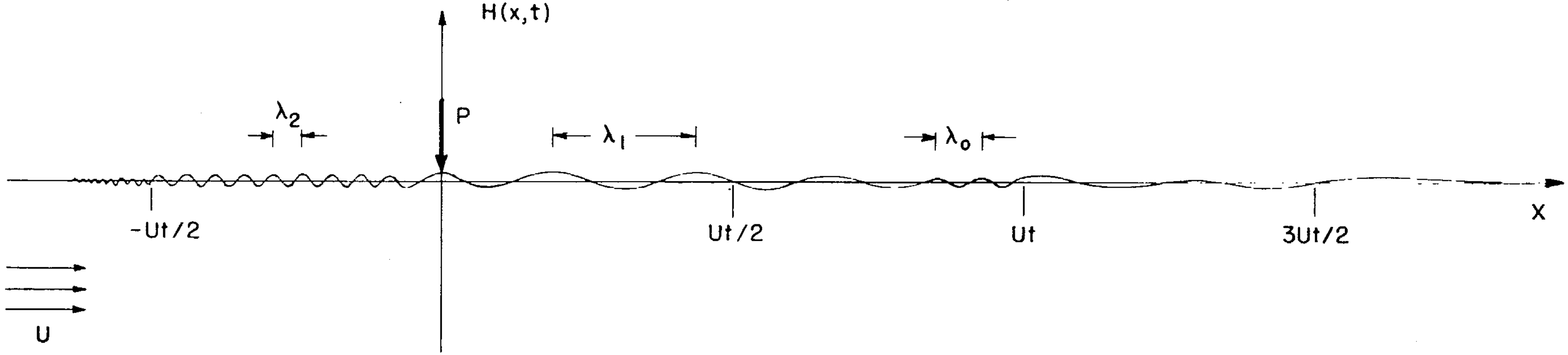


Fig. 4. The space distribution of the wave trains for large time.

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